INDIAN STATISTICAL INSTITUTE

Students' Brochure
PART II

Master of Mathematics

(Effective from 2021-22 Academic Year)

(See PART I for general information, rules and regulations)

The Headquarters is at
203 BARRACKPORE TRUNK ROAD
KOLKATA 700108

INDIAN STATISTICAL INSTITUTE
Master of Mathematics

Contents

1 Curriculum

All the courses listed below are allocated **three** lecture sessions and one practical/tutorial session per week. The practical/tutorial session consists of **one** period, which is meant to be used for discussion on problems, practicals, computer outputs, assignments, for special lectures and self study, etc. All these need not be contact hours.

First Year

Semester I		Semester II	
C1:	Measure Theory	C6:	Complex Analysis
C2:	Linear Algebra	C7:	Functional Analysis
C3:	Algebra I	C8:	Topology II
C4:	Topology I	C9:	Algebra II
C5:	Analysis of Several Variables	C10:	Number Theory

Second Year

Semester III		Semester IV	
C11:	Differential Geometry I	C13:	Partial Differential Equations
C12:	Probability Theory		Elective 4
	Elective 1		Elective 5
	Elective 2		Elective 6
	Elective 3/Project I		Elective 7/Project II

The list of permitted elective courses is given in Section 2. In addition, elective courses can also be chosen from the list of courses from other programmes given in Section 3. At most two elective courses can be chosen from the courses mentioned in Section 3.

In each of the last two semesters, a student can take a project in place of an elective course subject to the criteria mentioned in Section 4.

2 Elective Courses

A student has to choose elective courses offered from the following list. Not all of these courses can be offered in a particular semester.

E1: Fourier Analysis

E2: Differential Topology

E3: Graph Theory and Combinatorics

E52: Weak Convergence and Empirical Processes

Apart from the elective courses listed above, a student may also take elective courses in special topics from the following subjects: Algebra, Analysis, Number Theory, Topology, Geometry, Complex Analysis, Functional Analysis, Probability Theory, Harmonic Analysis, Algebraic Geometry, Algebraic Topology, Differential Topology, Differential Geometry, Commutative Algebra, Affine Algebraic Geometry, Discrete Mathematics, Statistics. The course-name should specify the topic (i.e., the subarea). For instance, "Special Topics in Functional Analysis: Operator Algebra" or "Special Topics in Algebra: Advanced Field Theory". A student may take more than one course in any of the above subjects, but the specific topic cannot be repeated. For instance, a student may take a course titled "Special Topics in Functional Analysis: Operator Algebras" and another course titled "Special Topics in Functional Analysis: Noncommutative Geometry" but the student cannot take more than one course titled "Special Topics in Functional Analysis: Noncommutative Geometry".

Special topics in areas of mathematics not mentioned in the above list may be offered after approval from the Dean/Associate Dean.

3 List of permissible elective courses from non-M.Math. programmes

At most two elective courses may be opted from the following list in the entire second year. The availability of a course in a specific year or semester may vary, depending on resources and requirements of other programmes (M.Stat./ MSQE/ M.Tech.(CS) etc.). The pre-requisites for these courses are as given in the corresponding syllabus documents and the students should take these courses after discussing with the instructor.

3.1 Jointly with M.Tech.(CS)

1. Computational Complexity
2. Combinatorial Geometry
3. Cryptology
4. Information and Coding Theory
5. Logic for Computer Science
6. Quantum Information Processing and Quantum Computation

3.2 Jointly with M.S. in Q.E.

1. Game Theory I

3.3 Jointly with M.Stat.

1. Analysis of Directional Data
2. Branching Processes
3. Life Testing and Reliability
4. Markov Processes and Martingale Problems
5. Mathematical Biology
6. Percolation Theory
7. Random walks and Electrical Networks
8. Resampling Techniques
9. Robust Statistics
10. Signal and Image Processing
11. Statistical Methods in Demography
12. Statistical Methods in Epidemiology and Ecology
13. Theory of Extremes and Point Processes
14. Theory of Games and Statistical Decisions [1]
15. Theory of Random Graphs

3.4 B.Math. electives

1. Games, Graphs and Algebra
2. Information Theory
3. Methods from Statistical physics.
4. Mathematics of Data Science

[1] This cannot be taken if the elective "Game Theory I" from MSQE (see (1) in 3.2) has been taken already.

5. Geometric Group Theory.
6. Geometric Algebra.
7. Computational Biology.
8. Introduction to Statistical Physics.
9. Quantum Computation and Quantum Information.

4 Rules concerning M.Math. projects

A student can take at most two projects during M.Math. Second year with at most one project in a semester. The project must be in Mathematics or related areas with sufficient Mathematical content.

4.1 Eligibility

A student should have the necessary background knowledge for taking a particular project. A student with an aggregate of 70% in previous semesters and having no compulsory back-paper can apply for a project. However, obtaining 70% in aggregate does not automatically qualify a student to undertake a project. A project will be undertaken by a student only if it is approved by the Teachers' committee subject to the guideline given below.

4.2 Evaluation of the project proposal

To evaluate the merit of a project proposal the teachers committee may consult experts from related areas. The teachers committee will ensure that:

(a) The student has the requisite background and motivation to undertake a particular project.

(b) The project has sufficient Mathematical content.

4.3 Evaluation of projects

There will be a core committee of faculty members responsible for evaluation of projects. The student will make two presentations, one around the time of the mid-semestral examination and the other around the time of the semestral examination, and submit two reports after their presentation. A committee consisting of 3 members, the supervisor, one member from the core committee and a subject expert, will evaluate the student on the basis of the presentation, a viva-voce examination and project reports. The respective weights are 40% (for the mid-semester presentation) and 60% (for the presentation at the end of the semester). For every evaluation, the presentation and the report will carry equal weights.

In case a student obtains less than 45% in the composite score then (s)he will be offered an opportunity to appear for the backpaper examination. The student should submit a revised project report by the last working day before the backpaper examination for the M. math. There will be project presentation during the week of the backpaper examination (or the week after that). The date will be finalized by the supervisor in consultation with the examiners and be conveyed to the Dean of studies. The scoring will be based only on the new presentation and the revised report. The maximum score possible will be 45%. The other rules and regulations regarding backpaper examination for a regular course will also apply.

4.4 Continuation

To take up project II in the fourth semester the student should satisfy the eligibility criteria as given above. However, if a student's aggregate is less than 70% but not less than 65% then based on recommendation of the supervisor the student may be allowed to continue with the same project in the 4th semester.

4.5 Application procedure

The interested student will have to apply for taking up a project, specifying the title of the project, the name of the supervisor and an abstract of the proposed work, within a date suggested by the Dean's office/teachers committee.

5 Detailed Syllabi of the Courses

5.1 Compulsory Courses

5.1.1 First Year, Semester I

C1: Measure Theory

- The concept of σ-algebra, Borel subsets of $\mathbb{R}$, Construction of Lebesgue and Lebesgue-Stieltjes measures on the real line following outer measure.
- Abstract measure theory: definition and examples of measure space, measurable functions, Lebesgue integration, convergence theorems (Fatou's Lemma, Monotone convergence and dominated convergence theorem).

- Caratheodory extension theorem, completion of measure spaces.
- Product measures and Fubini's theorem.
- L^p-spaces, Riesz-Fischer Theorem, approximation by step functions and continuous functions.
- Absolute continuity, Hahn-Jordan decomposition, Radon-Nikodym theorem, Lebesgue decomposition theorem. Functions of bounded variation.
- Complex measures.

If time permits: Vitali covering lemma, differentiation and fundamental theorem of calculus.

References

(a) H. L. Royden and Patrick Fitzpatrick: Real Analysis, Pearson, 4th edition.

(b) Robert B. Ash and Catherine A. Doleans-Dade, Probability and measure theory, GTM(211), Academic Press, 2nd edition.

(c) Elias M. Stein, Rami Shakarchi, Real Analysis: Measure Theory, Integration and Hilbert Spaces , Princeton Lectures in Analysis.

(d) Gerald B. Folland, Real Analysis: Modern Techniques and Their Applications, Pure and Applied Mathematics, A Wiley Series.

(e) G. de Barra, Measure Theory and Integration.

C2: Linear Algebra

- Quick review of solutions of a system of linear equations, vector spaces, subspaces, linear independence and span, Zorn's lemma and existence of basis, quotient spaces and direct sum of vector spaces, exact sequences and splittings, linear maps and matrices, matrix of a linear map in a basis, invertibility, rank and determinant, linear functionals, dual space, annihilator, transpose of a linear map.

  [This part is mostly a review and should be covered quickly with emphasis on problem solving.]
- Eigenvalues, algebraic and geometric multiplicities, characteristic and minimal polynomials, upper triangularization, diagonalizability and semisimplicity, decomposition into nilpotent and semisimple matrices, Cayley-Hamilton Theorem.

- Tensor product of vector spaces, extension of scalars, complexification, tensor product of linear maps, symmetric and exterior algebra, determinant as a multilinear map and Laplace expansion.
- Inner-product spaces, orthogonality, Gram-Schmidt orthogonalization, Bessel's inequality, projection and orthogonal projection, symmetric and Hermitian operators, orthogonal and unitary diagonalizability, normal operators, spectral theorem, bilinear and quadratic forms, positive definite operator, square-root of a positive operator, polar decomposition, isometry, rigid motions, the rotation group.
- Structure theory of finitely generated modules over PID and application to canonical forms.

References

(a) D.S. Dummit and R.M. Foote, Abstract Algebra, John Wiley (Asian reprint 2003).

(b) S. Lang, Algebra, GTM (211), Springer (Indian reprint 2002).

(c) K. Hoffman and R. Kunze, Linear Algebra, Prentice-Hall of India (1998).

(d) N.S. Gopalakrishnan, University Algebra, Wiley Eastern (1986).

(e) A. R. Rao and P. Bhimasankaram, Linear Algebra, TRIM(19), Hindustan Book Agency (2000).

(f) P. R. Halmos, Finite-Dimensional Vector Spaces: Second Edition.

C3: Algebra I

- Commutative rings with unity: examples, ring homomorphisms, ideals, quotients, isomorphism theorems with applications to non-trivial examples. Prime and maximal ideals, Zorn's Lemma and existence of maximal ideals. Product of rings, ideals in a finite product, Chinese Remainder Theorem. Prime and maximal ideals in a quotient ring and a finite product. Field of fractions of an integral domain. Irreducible and prime elements; PID and UFD.
- Polynomial Ring: universal property; division algorithm; roots of polynomials. Gauss's Theorem (R UFD implies R[X] UFD); irreducibility criteria. Symmetric polynomials: Newton's Theorem. Power Series.
- Modules over commutative rings: examples. Basic concepts: submodules, quotients modules, homomorphisms, isomorphism theorems, generators, annihilator, torsion, direct product and sum, direct summand, free modules, finitely generated modules, exact and split exact sequences.

- Noetherian rings and modules, algebras, finitely generated algebras, Hilbert Basis Theorem. Tensor product of modules: definition, basic properties and elementary computations. Time permitting, introduction to projective modules.

References

(a) D.S. Dummit and R.M. Foote, Abstract Algebra, John Wiley (Asian reprint 2003).

(b) N. Jacobson, Basic Algebra Vol. I, W.H. Freeman and Co (1985).

(c) S. Lang, Algebra, GTM (211), Springer (Indian reprint 2004).

(d) N.S. Gopalakrishnan, University Algebra, Wiley Eastern (1986).

(e) N.S. Gopalakrishnan, Commutative Algebra (Chapter 1), Oxonian Press (1984).

C4: Topology I

- Topological spaces, open and closed sets, basis, closure, interior and boundary. Subspace topology, Hausdorff spaces. Continuous maps: properties and constructions; Pasting Lemma. Homeomorphisms. Product topology.
- Connected, path-connected and locally connected spaces. Lindelöf and Compact spaces, Locally compact spaces, one-point compactification and Tychonoff's theorem. Paracompactness and Partitions of unity (if time permits).
- Countability and separation axioms., Urysohn embedding lemma and metrization theorem for second countable spaces. Urysohn's lemma, Tietze extension theorem and applications. Complete metric spaces. Baire Category Theorem and applications.
- Quotient topology: Quotient of a space by a subspace. Group action, Orbit spaces under a group action. Examples of Topological Manifolds.
- Topological groups. Examples from subgroups of $GL_n(\mathbb{R})$ and $GL_n(\mathbb{C})$.
- Homotopy of maps. Homotopy of paths. Fundamental Group.

References

(a) J. R. Munkres, Topology: a first course, Prentice-Hall (1975).

(b) G.F. Simmons, Introduction to Topology and Modern Analysis, TataMcGraw-Hill (1963).

(c) M.A. Armstrong, Basic Topology, Springer.

(d) J.L. Kelley, General Topology, Springer-Verlag (1975).

(e) J. Dugundji, Topology, UBS (1999).

(f) I. M. Singer and J. A. Thorpe, Lecture notes on elementary topology and geometry, UTM, Springer.

C5: Analysis of Several Variables

- Metric Topology of $\mathbb{R}^n$. Topology induced by l_p norms ($p = 1, 2, \infty$) on $\mathbb{R}^n$ and their equivalence. Continuous functions on $\mathbb{R}^n$. Separation properties. Compact subsets of $\mathbb{R}^n$. Path-connectivity. Topological properties of subgroups like $GL_n(\mathbb{R})$, $GL_n(\mathbb{C})$, $O(n)$, Hilbert-Schmidt norm and operator norm on $M_n(\mathbb{R})$. Sequence and series in $M_n(\mathbb{R})$. Exponential of a matrix.
- Differentiation and integration of functions on $\mathbb{R}^n$. Partial derivatives of real-valued functions on $\mathbb{R}^n$. Differentiability of maps from $\mathbb{R}^m$ to $\mathbb{R}^n$ and the derivative as a linear map. Jacobian theorem. Chain Rule. Mean value theorem. Higher derivatives and Schwarz theorem, Taylor expansions in several variables. Inverse function theorem and implicit function theorems. Local maxima and minima, Lagrange multiplier method.
- Vector fields on $\mathbb{R}^n$. Integration of vector fields and flows. Picard's Theorem.
- Riemann integration of bounded real-valued functions on rectangles (product of intervals). Existence of the Riemann integral for sufficiently well-behaved functions. Iterated integral and Fubini's theorem. Brief treatment of multiple integrals on more general domains. Change of variable and the Jacobian formula.
- Differential forms on $\mathbb{R}^n$. Wedge product of forms. Pullback of differential forms. Exterior differentiation of forms. Integration of compactly supported n-forms on $\mathbb{R}^n$. Change of variable formula revisited. Integration of k-forms along singular k-chains in $\mathbb{R}^n$. Stokes' theorem on chains. [Special emphasis on curves and surfaces in $\mathbb{R}^2$ and $\mathbb{R}^3$. Line integrals, Surface integrals. Gradient, Curl and Divergence operations, Green's theorem and Gauss's (Divergence) theorem.]

Reference Texts:

(a) M. Spivak: Calculus on manifolds, Benjamin (1965).

(b) T.Apostol: Mathematical Analysis. S. Lang, Algebra, GTM (211), Springer (Indian reprint 2002).

(c) K. Mukherjea: Differential Calculus in Normed Linear Spaces

5.1.2 First Year, Semester II

C6: Complex Analysis

- Review of sequences and series of functions including power series, Complex differentiation and Cauchy-Riemann equation, Cauchy's theorem and Cauchy's integral formula, Power series expansion of holomorphic function, zeroes of holomorphic functions, Maximum Modulus Principle, Liouville's Theorem, Morera's Theorem.
- Complex logarithm and winding number, Singularities, Meromorphic functions, Casorati Weierstrass theorem, Riemann sphere, Laurent series, Residue Theorem and applications to evaluation of definite integrals, Open Mapping Theorem, Rouche's Theorem.
- Conformal maps, Schwarz lemma, Linear fractional transformations, automorphisms of a disc, Introduction to Gamma function.
- Equicontinuity and Arzela-Ascoli Theorem, Normal family, Montel's theorem and Riemann mapping theorem.
- If time permits then the following topics can also be covered: Mittag-Leffler Theorem, Infinite product, Weierstrass factorization theorem.

References

(a) Complex Analysis- L. Ahlfors.

(b) Elementary Theory of Analytic Functions of One or Several Complex Variables- H. Cartan

(c) Complex Analysis- E. M. Stein, R. Shakarchi.

(d) Complex Analysis- D. Sarason

C7: Functional Analysis

- Quick review of sequences and series of functions, equicontinuity, Arzela-Ascoli theorem.
- Normed linear spaces and Banach spaces. Bounded linear operators. Dual of a normed linear space. Hahn-Banach theorem, uniform boundedness principle, open mapping theorem, closed graph theorem. Computing the dual of some well known Banach spaces. Weak and weak-star topologies, Banach-Alaoglu Theorem. The double dual. L^p-spaces and their duality, Weierstrass and Stone-Weierstrass Theorems.

- Hilbert spaces, adjoint operators, self-adjoint and normal operators, spectrum, spectral radius, analysis of the spectrum of a compact operator on a Banach space, spectral theorem for compact self-adjoint operators on Hilbert spaces. Basics of complex measures and statement of the Riesz representation theorem for the space $C(X)$ for a compact Hausdorff space X.
- If time permits, some of the following topics may be covered: Sketch of proof of the Riesz Representation Theorem for $C(X)$, Goldstein's Theorem, reflexivity; spectral theorem for bounded normal operators.

References

(a) Real and complex analysis, W. Rudin, McGraw-Hill (1987).

(b) Functional analysis, W. Rudin, McGraw-Hill (1991).

(c) A course in functional analysis, J. B. Conway, GTM (96), Springer-Verlag (1990).

(d) Functional analysis, K. Yosida, Grundlehren der Mathematischen Wissenschaften (123), Springer-Verlag (1980).

C8: Topology II

- Review of fundamental groups, necessary introduction to free product of groups, Van Kampen's theorem. Covering spaces, lifting properties, Universal cover, classification of covering spaces, Deck transformations, properly discontinuous action, covering manifolds, examples.
- Categories and functors. Simplicial homology. Singular homology groups, axiomatic properties, Mayer-Vietoris sequence, homology with coefficients, statement of universal coefficient theorem for homology, simple computation of homology groups.
- CW-complexes and Cellular homology, Simplicial complex and simplicial homology as a special case of Cellular homology, Relationship between fundamental group and first homology group. Computations for projective spaces, surfaces of genus g.

References

(a) A. Hatcher, Algebraic Topology, Cambridge University Press (2002).

(b) W. S. Massey, A basic course in algebraic topology, GTM (127), Springer (1991).

(c) J. R. Munkres, Topology: a first course, Prentice-Hall (1975).

(d) J. R. Munkres, Elements of algebraic topology, Addison-Wesley (1984).

(e) M. J. Greenberg, Lectures on algebraic topology, Benjamin (1967).

(f) I. M. Singer and J. A. Thorpe, Lecture notes on elementary topology and geometry, UTM, Springer.

(g) E. Spanier, Algebraic Topology, Springer-Verlag (1982).

C9: Algebra II

- Review of normal subgroups, quotient, isomorphism theorems, Group actions with examples, class equations and their applications, Sylow's Theorems; groups and symmetry. Direct sum and free Abelian groups. Composition series, exact sequences, direct product and semidirect product with examples. Results on finite groups: permutation groups, simple groups, solvable groups, simplicity of A_n.
- Algebraic and transcendental extensions; algebraic closure; splitting fields and normal extensions; separable, inseparable and purely inseparable extensions; finite fields. Galois extensions and Galois groups, Fundamental theorem of Galois theory, cyclic extensions, solvability by radicals, constructibility of regular n-gons, cyclotomic extensions.
- Time permitting: Topics from Trace and Norms, Hilbert Theorem 90, Artin-Schreier theorem, Transcendental extensions, Real fields.

References

(a) J.J. Rotman, An Introduction to the theory of groups, GTM (148), Springer-Verlag (2002).

(b) D.S. Dummit and R.M. Foote, Abstract Algebra, John Wiley (Asian reprint 2003).

(c) S. Lang, Algebra, GTM (211), Springer (Indian reprint 2004).

(d) N.S. Gopalakrishnan, University Algebra, Wiley Eastern (1986).

(e) N. Jacobson, Basic Algebra, W.H. Freeman and Co (1985).

(f) G. Rotman, Galois theory, Springer (Indian reprint 2005).

(g) TIFR pamphlet on Galois theory.

(h) Patrick Morandi, Field and Galois Theory, GTM(167) Springer-Verlag (1996).

(i) M. Nagata, Field theory, Marcel-Dekker (1977).

C10: Number Theory

- Brief review of Division Algorithm, gcd and lcm, Euclidean algorithm; Linear Diophantine equations, congruences and residues, the Chinese Remainder Theorem; The ring $\mathbb{Z}/n\mathbb{Z}$ and its group of units, The Euler ϕ-function, Fermat's little theorem, Euler's theorem, Wilson's Theorem, Sums of two and four squares.
- Pythagorean triplets and their geometric interpretation (rational points on circles); Rational points on general conics; Fermat's method of infinite descent and application to simple Diophantine equations like $x^4 + y^4 = z^2$; The Hasse principle for conics (statement only), Brief discussion on rational points on cubics and the failure of the Hasse principle (statement only).
- Polynomial congruences and Hensel's Lemma; Quadratic residues and non-residues, Euler's criterion, Detailed study of the structure of the group of units of $\mathbb{Z}/n\mathbb{Z}$, Primitive roots; Definition and properties of the Legendre symbol, Evaluation of Gauss sums, The law of quadratic reciprocity for Legendre symbols; Extension to the Jacobi symbols.
- Arithmetical functions and their convolutions, multiplicative and completely multiplicative functions, examples like the divisor function $d(n)$, the Euler function $\phi(n)$, the Möbius function $\mu(n)$ etc.; The Möbius inversion formula; Sieve of Eratosthenes; Notion of order of magnitude and asymptotic formulae; Euler and Abel summation formulae, Hyperbola method of Dirichlet, Average order of magnitude of various arithmetical functions such as $\phi(n), d(n)$ etc.; Statement of the Prime Number Theorem; Elementary estimates due to Chebyshev and Mertens on primes.
- Algebraic integers, Arithmetic in $\mathbb{Z}[i]$ and $\mathbb{Z}[\omega]$, Primes of the forms $x^2 + y^2$ and $x^2 + xy + y^2$; Integers in quadratic number fields; Examples of failure of unique factorization; Units in the ring of integers of a real quadratic field and application to the Brahmagupta-Pell equation.
- One or more topics from the following list can be discussed if time permits:
  A. Absolute values on $\mathbb{Q}$, Ostrowski's Theorem, Completions of $\mathbb{Q}$, $\mathbb{Q}_p$ and $\mathbb{Z}_p$, The p-adic topology.
  B. The notion of algebraic and transcendental numbers; Transcendence of e, Diophantine Approximation, Dirichlet's Theorem; Liouville's Theorem, Statement of Roth's Theorem.

C. Continued fractions; Applications to Diophantine approximation, Application to the Brahmagupta-Pell equation.
D. Introduction to Geometry of numbers.
E. Application of Number Theory to RSA and other cryptosystems.

Reference Texts:

(a) T. Apostol, Analytic Number Theory

(b) D. Burton, Elementary Number Theory

(c) G.H Hardy and E.M. Wright, An Introduction to The Theory of Numbers

(d) K. Ireland, M. Rosen, A Classical Introduction to Modern Number Theory

(e) Manin and Panchishkin, Introduction to Modern Number Theory

(f) S.J. Miller and R. Takloo-Bighash, An Invitation to Modern Number Theory

(g) I. Niven, H. S. Zuckerman, H. L. Montgomery, The Theory of Numbers

(h) J.-P. Serre, A Course in Arithmetic.

5.1.3 Second Year, Semester III

C11: Differential Geometry I

- *Smooth manifolds*: Manifolds in $\mathbb{R}^n$, submanifolds, manifolds with boundary. Smooth maps between manifolds. Regular values. Examples of manifolds: A) Curves and surfaces in $\mathbb{R}^2$ and $\mathbb{R}^3$. B) Level surfaces in $\mathbb{R}^{n+1}$, C) Inverse image of regular values. Tangent spaces, derivatives of smooth maps, smooth vector fields, Existence of integral curves of a vector field near a point.
- *Geometry of curves and surfaces*: Parametrized curves in $\mathbb{R}^3$, length, integral formula for smooth curves, regular curves, parametrization by arc length. Osculating plane of a space curve, Frenet frame, Frenet formula, curvature, invariance under isometry and reparametrization. Discussion of the cases for plane curves, rotation number of a closed curve, osculating circle, 'Umlaufsatz'.
- *Surfaces in* $\mathbb{R}^3$: Existence of a normal vector of a connected surface. Gauss map. The notion of a geodesic on a surface. The existence and uniqueness of a geodesic on a surface through a given point with a given velocity vector thereof. Covariant derivative of a smooth vector field. Parallel vector field along a curve. Existence and uniqueness theorem of a parallel vector field along a curve with a given initial vector. The Weingarten map of a surface at a point,

its self-adjointness property. Normal curvature of a surface at a point in a given direction. Principal curvatures, first and second fundamental forms, Gauss curvature and mean curvature. Gauss-Bonnet theorem (statement only).

- *Differential forms and orientation*: Differential Forms, Orientation of manifolds, Integration of forms, Stokes' Theorem. (proof to be given if time permits). Proof of Gauss-Bonnet theorem (if time permits).

References

(a) B O' Neill, Elementary Differential Geometry, Academic Press (1997).

(b) A. Pressley, Elementary Differential Geometry, Springer (Indian Reprint 2004).

(c) J. A. Thorpe, Elementary topics in Differential Geometry, Springer (Indian reprint, 2004).

(d) V. Guillemin and A. Pollack, Differential Topology, Prentice-Hall.

C12: Probability Theory

- Revision of Measure theory: probability spaces, distributions, random variables, standard random variable examples, expected value, inequalities (Holder, Cauchy-Schwarz, Jensen, Markov, Chebyshev) , convergence notions(convergence in probability and almost sure, L^p), application of DCT, MCT, Fatou with examples, Revision of Fubini's theorem.
- Independence, sum of random variables, constructing independent random variables, weak law of large numbers, Borel-Cantelli lemmas, First and Second Moment methods, Chernoff bounds and some applications.
- Strong law of large number, Kolmogorov $0-1$ law. Convergence of random series. Kolmogorov's three series theorem.
- Weak convergence, tightness, characteristic functions with examples, Central limit theorem (iid sequence and triangular array).

References

(a) Rick Durrett: Probability (Theory and Examples).

(b) Patrick Billingsley: Probability and measure.

(c) Robert Ash: Basic probability theory.

(d) Leo Breiman: Probability.

(e) David Williams: Probability with Martingales.

5.1.4 Second Year, Semester IV

C13: Partial Differential Equations

- Basics of ODE: (local as well as global) existence and uniqueness results, Picard iteration, Gronwall's inequality, solving some first order and second order equations. [7 lectures]
- Introduction to PDE: order of a PDE, classification of PDEs into linear, semi-linear, quasi-linear, and fully nonlinear equations, Examples of equations from Physics, Geometry, etc. The notion of well-posed PDEs [1 lecture]
- First order PDEs: Method of characteristics, existence and uniqueness results of the Cauchy problem for quasilinear and fully nonlinear equations. [9 lectures] Second order linear PDEs in two independent variables: classification into hyperbolic, parabolic and elliptic equations, canonical forms. [1 lecture]
- Laplace equation: Definition of Harmonic functions. Mean-value property, Strong Maximum principle for harmonic functions, Liouville's theorem, smoothness of harmonic functions, Poisson's formula. Harmonic functions in rectangles, cubes, circles, wedges, annuli. [9 lectures]
- Heat equation: Fundamental solution of heat equation, Duhamel's principle, weak and strong maximum principles, smoothness of solutions of heat equation, ill-posedness of backward heat equation. [9 lectures]
- Wave equation: well-posedness of initial and boundary value problem in 1D and d'Alembert formula. method of descent in 2D and 3D. Duhamel's principle, domain of dependence, range of influence, finite speed of propagation. [8 lectures]
- Boundary problems: Separation of variables, Dirichlet, Neumann and Robin conditions. The method of seperation of variables for Laplace, Heat and Wave equations. [2 lectures]

References

(a) Evans, L. C. Partial Differential Equations, AMS, 2010.

(b) Han, Q. A Basic Course in Partial Differential Equations, AMS, 2011.

(c) McOwen, R. Partial Differential Equations: Methods and Applications, Pearson, 2002.

(d) Pinchover, Y. and Rubinstein, J. An Introduction to Partial Differential Equations, Cambridge, 2005.

(e) Fritz John Partial Differential Equations, Springer.

(f) Partial Differential Equations: An introduction by Walter Strauss.

5.2 Elective Courses

E1: Fourier Analysis

- Fourier Series on $\mathbb{T}$:

  a) Dirichlet problem for the unit disc and origin of Fourier series, continuity of translation on $L^p(\mathbb{T})$ and elementary convolution inequalities, approximate identity.

  b) Fourier series and its elementary properties, completeness of trigonometric polynomials and Riemann-Lebesgue lemma, Uniqueness of Fourier coefficients and the Fourier inversion, Plancherel theorem, Weyl's equidistribution theorem.

  c) Dirichlet kernel and pointwise convergence of Fourier series for Lipschitz continuous functions, Riemann's localization principle, existence of a continuous function with divergent Fourier series.

  d) Cesaro and Abel summability, Poisson integral and solution of Dirichlet problem for appropriate function classes.

  e) Marcinkiewicz interpolation theorem, Young's inequality and Hausdorff-Young inequality, norm convergence of Fourier series for L^p, $1 < p < \infty$.

- Fourier transform on $\mathbb{R}^d$:

  a) Elementary properties of the Fourier transform involving translation, dilation, rotation, decay and smoothness, Riemann Lebesgue lemma, Fourier transform of Gaussian and the Poisson kernel, the Fourier inversion. Schwartz class functions and its image under Fourier transform.

  b) Fourier transform of L^2-functions and the Plancherel theorem, Hausdorff-Young inequality, Paley-Wiener theorem, Poisson summation formula.

  c) Tempered distribution and its Fourier transform, computation of some distributional Fourier transform.

  d) Weak L^p spaces, Method of maximal function, Lebesgue differentiation theorem, almost everywhere convergence of Poisson integrals.

  e) Nontangential convergence of Poisson integral and characterization of Poisson integral of L^p functions (If time permits).

References

(a) Real and complex analysis- W. Rudin.
(b) Topics in Functional analysis- S. Kesavan.
(c) Introduction to Fourier Analysis on Euclidean spaces- E. M. Stein, G. Weiss.
(d) Fourier Analysis- E. M. Stein, R. Shakarchi.
(e) Introduction to Harmonic Analysis-Y. Katznelson

E2: Differential Topology

- Manifolds in $\mathbb{R}^n$, submanifolds, smooth maps of manifolds, derivatives and tangents, Inverse function theorem and immersions, submersions, Transversality, Homotopy and stability, Sard's theorem and Morse functions, embedding manifolds in Euclidean space.
- Intersection of transverse manifolds, mod 2 intersection theory and winding numbers. Jordan-Brouwer separation theorem.
- Orientation of manifolds, Oriented intersection number. Lefschetz fixed point theory. Hopf Degree theorem.

References

(a) V. Guillemin and A. Pollack, Differential Topology, Prentice-Hall.
(b) J. W. Milnor, Topology from the Differentiable Viewpoint, Univ Press of Virginia (1965).
(c) R. Bott and L. W. Tu, Differential forms in algebraic topology, GTM (82), Springer Verlag (1982).

E3: Graph Theory and Combinatorics

(*Note: This course may be combined with the Discrete Mathematics course of M.Stat. or M. Tech CS.*)

- Construction and Uniqueness of Finite Fields, Linear Codes, Macwilliams identity, Finite projective planes, strongly regular graphs and regular 2-graphs. t-designs with emphasis on Mathieu designs. Counting arguments and inclusion-exclusion principle. Ramsey Theory: graphical and geometric.

- Graphs and digraphs, connectedness, trees, degree sequences, connectivity, Eulerian and Hamiltonian graphs, matchings and SDR 's, chromatic numbers and chromatic index, planarity, covering numbers, flows in networks, enumeration, Inclusion-exclusion, Ramsey's theorem, recurrence relations and generating functions.
- Time permitting, some of the following topics may be done: (i) strongly regular graphs, root systems, and classification of graphs with least eigenvalue, (ii) Elements of coding theory (Macwilliams identity, BCH, Golay and Goppa codes, relations with designs).

References

(a) F. Harary, Graph Theory, Addision-Wesley (1969), Narosa (1988).

(b) D. B. West, Introduction to Graph Theory, Prentice-Hall (Indian Edition 1999).

(c) J. A. Bondy and U. S. R. Murthy, Graph Theory and Applications, MacMillan (1976).

(d) H. J. Ryser, Combinatorial Mathematics, Carus Math. Monograph, MAA (1963).

(e) M. J. Erickson, Introduction to Combinatorics, John Wiley (1996).

(f) L. Lovasz, Combinatorial Problems and Exercises, AMS Chelsea (1979).

E4: Commutative Algebra I

- Quick review of Rings and ideals: ideals in quotient rings; prime ideals under quotient, existence of maximal ideals; operations on ideals (sum, product, quotient and radical); ideals and prime ideals in product rings, Chinese Remainder theorem; nilradical and Jacobson radical; extension and contraction of ideals under ring homomorphisms;
- Free modules; Projective Modules; Tensor Product of Modules and Algebras; Flat, Faithfully Flat and Finitely Presented Modules; Shanuel's Lemma.
- Localisation and local rings, universal property of localisation, extended and contracted ideals and prime ideals under localisation, localisation and quotients, exacteness property, Nagata's criterion for UFD and applications.
- Prime avoidance. Results on prime ideals like theorems of Cohen and Isaac, equivalence of PID and one-dimensional UFD.
- Modules over local rings. Cayley-Hamilton, NAK lemma and applications. Examples of local-global principles. Projective and locally free modules. Patching up of Localisation.

- Polynomial and Power Series Rings. Noetherian Rings and Modules. Hilbert's Basis Theorem, Graded Rings, equivalence of Noetherian rings and finitely generated algebras for graded rings.
- Integral Extensions: integral closure, normalisation and normal rings. Cohen-Seidenberg Going-Up Theorem. Hilbert's Nullstellensatz and applications. Introduction to Valuation rings.
- Time permitting: Introduction to Grobner basis.

References

(a) N.S. Gopalakrishnan, Commutative Algebra, Oxonian Press (1984).

(b) M.F. Atiyah and I.G. Macdonald, Introduction to commutative algebra, Addision-Wesley(1969).

(c) M. Reid: Undergraduate commutative algebra, LMS Student Texts (29), Cambridge Univ. Press (1995).

(d) R.Y. Sharp: Steps in commutative algebra, LMS Student Texts, Cambridge Univ. Press.

(e) E. Kunz: Introduction to commutative algebra and algebraic geometry, Birkhauser (1985).

(f) D.S. Dummit and R.M. Foote: Abstract Algebra (Part V), John Wiley (2003).

(g) D. Eisenbud: Commutative algebra with a view toward algebraic geometry GTM (150), Springer-Verlag (1995).

E5: Representation Theory of Finite Groups

- Representation of finite groups - definition and examples; symmetric and exterior powers; group algebra, Maschke's Theorem, Simple Modules over group algebras.
- Characters and Orthogonality relations; Fourier analysis on finite Abelian groups; Burnside's pq theorem. Construction of character tables with examples.
- Permutation representations; Induced representations; Frobenius reciprocity;
- Representation theory of symmetric groups – partitions and tableau, Young diagrams and Frobenius character formula (from Fulton-Harris).
- Time permitting: Brauer's theorem on induced characters.

References

(a) Benjamin Steinberg, Representation theory of finite groups, Springer (2012).

(b) E. Kowalski, An Introduction to the Representation Theory of Groups, AMS (2014).

(c) W. Fulton And J. Harris, Representation Theory, A first course; GTM 129 Springer (1991).

(d) C.W. Curtis and I. Reiner, Representation Theory of finite Groups and Associative Algebras, Springer.

(e) J-P Serre, Linear Representations of Finite Groups; GTM 42 Springer (2012).

(f) N. Jacobson, Basic Algebra II, W.H. Freeman and Co (1985).

E6: Lie Groups and Lie Algebras

- Linear Lie groups: the exponential map and the Lie algebra of linear Lie group, some calculus on a linear Lie group, invariant differential operators, finite dimensional representations of a linear Lie group and its Lie algebra. Examples of linear Lie group and their Lie algebras, e.g., Complex groups: $GL(n, \mathbb{C})$, $SL(n, \mathbb{C})$, $SO(n, \mathbb{C})$, groups of real matrices in those complex groups: $GL(n, \mathbb{R})$, $SL(n, \mathbb{R})$, $SO(n, \mathbb{R})$, Isometry groups of Hermitian forms $SO(m, n)$, $U(m, n)$, $SU(m, n)$. Finite dimensional representations of $su(2)$ and $SU(2)$ and their connection. Exhaustion using the lie algebra $su(2)$. [2 weeks]
- Lie algebras in general, Nilpotent, solvable, semisimple Lie algebra, ideals, Killing form, Lies and Engels theorem. Universal enveloping algebra and Poincare-Birkhoff-Witt Theorem (without proof). [6 weeks]
- Semisimple Lie algebra and structure theory: Definition of Linear reductive and linear semisimple groups. Examples of Linear connected semisimple/ reductive Lie groups along with their Lie algebras (look back at 2 above and find out which are reductive/semisimple). Cartan involution and its differential at identity; Cartan decomposition $g = k + p$, examples of k and p for the groups discussed above. Definition of simple and semisimple Lie algebras and their relation, Cartans criterion for semisimplicity. Statements and examples of Global Cartan decomposition, Root space decomposition; Iwasawa decomposition; Bruhat decomposition. [6 weeks]
- If time permits, one of the following topics: (i) A brief introduction to Harmonic Analysis on $SL(2, \mathbb{R})$. (ii) Representations of Compact Lie Groups and Weyl Character Formula. (iii) Representations of Nilpotent Lie Groups.

References

(a) J.E. Humphreys: Introduction to Lie algebras and representation theory, GTM (9), Springer-Verlag (1972).

(b) S.C. Bagchi, S. Madan, A. Sitaram and U.B. Tiwari: A first course on representation theory and linear Lie groups, University Press (2000).

(c) Serge Lang: $SL(2, \mathbb{R})$. GTM (105), Springer (1998).

(d) W. Knapp: Representation theory of semisimple groups. An overview based on examples, Princeton Mathematical Series (36), Princeton University Press (2001).

(e) B.C. Hall, Lie Groups, Lie Algebras and Representations: An Elementary Introduction, Springer (Indian reprint 2004).

E7: Algebraic Geometry

(*Note: For students opting for 'Algebraic Geometry', a prior knowledge of 'Commutative Algebra' is desirable.*)

- Topics from: Polynomial rings, Hilbert Basis theorem, Noether normalisation lemma, Hilbert Nullstellensatz, Affine and Projective spaces, Affine Schemes, Elementary dimension theory, Smoothness, Curves, Divisors on curves, Bezout's theorem, Abelian differential, Riemann Roch for curves.

References

(a) W. Fulton, Algebraic curves. An introduction to algebraic geometry, Addison-Wesley (1989).

(b) D.S. Dummit and R.M. Foote, Abstract Algebra (Part V), John Wiley.

(c) C.G. Gibson, Elementary Geometry of Algebraic Curves, Cambridge.

(d) I.R. Shafarevich, Basic algebraic geometry, Springer.

(e) J. Harris, Algebraic geometry. A first course, GTM (133), Springer-Verlag (1995).

(f) K. Kendig, Elementary algebraic geometry, GTM (44), Springer-Verlag (1977).

(g) D. Mumford, The Red Book of Varieties and Schemes, Springer.

(h) C. Musili, Algebraic geometry for beginners, TRIM (20), HBA (2001).

E8: Commutative Algebra II

- Associated Primes. Primary Decomposition. Homomorphisms and $\mathrm{Ass}\ M$. $\mathrm{Supp}\ M$, Artininan Modules. Modules of Finite Length.

- Going-up and Going-Down Theorems. Finiteness of Integral Closure. Krull-Akizuki theorem.
- Dimension Theory: Hilbert Samuel Polynomial. Krull's Principal Ideal Theorem. Dimension Theorem.
- Normalisation Lemma. Hilbert's Nullstellensatz. Integral Closure of Affine Domains.
- Valuations, Discrete Valuation Rings, Dedekind Domains, Fractional and Invertible Ideals, Ramification Formula.
- Results on Normal and Regular Rings: Local property of Normal Domains, Normality and DVR at height one primes, Intersection of DVRs; Jacobian criterion for regular local rings (of affine algebras).
- Graded Rings and Modules. Artin-Rees Lemma. I-adic filtrations. Completion. Exacteness and Flatness properties. Krull's Intersection Theorem. Hensel's Lemma and applications. Weierstrass's Preparation theorem.
- Time Permitting:
  A. Homological Characterisation of Regular Local Rings, Regular Local Ring is a UFD, Derivations and Modules of Differentials.
  B. Cohen-Macaulay rings, Auslander-Buchsbaum-Serre theorem, Gorenstein rings and duality.

References

(a) N.S. Gopalakrishnan, Commutative Algebra, Oxonian Press (1984).

(b) H. Matsumura, Commutative Algebra, W.A. Benjamin (1970).

(c) TIFR pamphlet on Homological Methods in Commutative Algebra.

(d) D. Eisenbud, Commutative algebra with a view toward algebraic geometry, GTM (150), Springer-Verlag (1995).

(e) J.P. Serre, Local Algebra, Springer-Verlag (2000).

(f) N. Bourbaki, Commutative Algebra, Springer-Verlag (1995).

(g) M. Reid, Undergraduate Commutative Algebra, LMS Student Texts (29), Cambridge Univ. Press (1995).

(h) B. Singh, Basic Commutative Algebra, World Scientific Publishing Company, (2011)

E9: Random Measures and Geometry

[Prerequisite: Probability Theory]

- Point processes and random measures, Poisson processes, Stationary point processes and random measures. Cox point processes. Mappings, Markings and Thinning. Mecke equation and Factorial measures. Palm distributions - Slivnyak-Mecke theorem, Local interpretation. Ergodic theorem. Boolean model, capacity functional and the random geometric graph. Perturbation analysis. Limit theorems in stochastic geometry.

References

(a) G. Last and M. Penrose: Lectures on the Poisson processes.

(b) R. Schneider and W. Weil: Stochastic and Integral Geometry.

(c) F. Baccelli and B. Blaszczyszyn: Stochastic Geometry and Wireless Networks. Volume I.

(d) O. Kallenberg: Random measures and applications.

E10: Modular Forms

- Elliptic functions, the Weierstrass $\wp$-function, Eisenstein series, the modular group, Fundamental domain for the modular group SL(2, $\mathbb{Z}$).
- Modular functions and modular forms for the modular group, Fourier expansion, Cusp forms, the Valence formula, the structure of the space modular forms, the j-function, Connection to elliptic curves, Dedekind η-function, Jacobi's product formula for the Ramanujan Δ-function.
- Bounds on the Fourier coefficients of cusp forms, Statement of the Ramanujan conjecture, L-function of modular forms, Functional equation.
- Hecke operators and Petersson inner product.
- Time permitting: Introduction to modular forms on congruence subgroups, Introduction to Maass forms etc.

References

(a) J.-P. Serrre: A Course in Arithmetic.

(b) N. Koblitz: Elliptic Curves and Modular Forms.

(c) H. Iwaniec: Topics in Classical Automorphic Forms.

E11: Analytic Number Theory

- Basic theory of multiplicative functions, Dirichlet convolutions, sums of multiplicative functions, Euler-Maclaurin summation formula, Dirichlet's hyperbola method, the Perron formula and Mellin inversion formula, Poisson summation formula.
- The Riemman zeta function, analytic continuation, the functional equation, non-vanishing on the line $\Re s = 1$, the zero-free region, the Prime Number Theorem.
- Dirichlet characters and Dirichlet L-functions, the functional equation, non-vanishing on the line $\Re s = 1$, Dirichlet's theorem on primes in arithmetic progression, zero-free region, the exceptional zero and the theorems of Landau and Siegel.
- Time permitting: Introduction to sieve methods, character sums, subconvex bounds of L-functions etc.

References

(a) H. Davenport: Multiplicative Number Theory.

(b) T. Apostol: Analytic Number Theory.

(c) H. Iwaniec and E. Kowalski: Analytic Number Theory.

E12: Sieve Methods

- The sieve of Eratosthenes-Legendre, the set-up of the sieve problem.
- Brun's pure sieve, The combinatorial sieve, Buchstab iterations, the fundamental lemma of sieve theory, applications to the Titchmarsh divisor problem Selberg sieve and applications.
- The large sieve and its applications.
- Time permitting: Bombieri-Vinogradov Theorem, Gaps between primes etc.

References

(a) H. Halberstam and H.E. Richert: Sieve Methods.

(b) J. Friedlander and H. Iwaniec: Opera de Cribro.

E13: Riemann Surfaces

- Definition of Riemann surfaces, holomorphic functions on Riemann surfaces, holomorphic maps between Riemann surfaces. Uniqueness principle. Local form of nonconstant holomorphic maps, local degree, open mapping theorem, maximum modulus principle. Meromorphic functions on Riemann surfaces, field of meromorphic functions. Covering surfaces. Statement of Uniformization Theorem. Automorphism groups of the Riemann sphere, the unit disk and the complex plane. Classification of Riemann surfaces with abelian fundamental group. Sheaf of holomorphic functions on a Riemann surface, germs of holomorphic functions, analytic continuation, Monodromy theorem. Branched coverings of Riemann surfaces induce finite extensions of function fields. Existence and uniqueness of the branched covering associated to a finite extension of function fields. Algebraic curves in the complex projective plane. Normalization theorem for algebraic curves. Isomorphism of group of deck transformations of a branched covering and Galois group of corresponding extension of function fields. Riemann-Hurwitz formula and application to computing genus. Almost complex structure, Hodge star operator, d-bar operator on a Riemann surface. Harmonic, holomorphic and meromorphic 1-forms on a Riemann surface, residues and Residue theorem. Hilbert space of square integrable 1-forms on a Riemann surface. Exact, co-exact and harmonic 1-forms. Weyl's lemma on harmonic functions. Hodge theorem for compact Riemann surfaces of genus g. Riemann bilinear relations for differentials of the 1st kind, period matrix, Jacobian variety, Abel-Jacobi map. Construction of harmonic and meromorphic 1-forms on Riemann surfaces with prescribed singularities. Riemann existence theorem. Mittag-Leffler problem for finding meromorphic functions and meromorphic 1-forms with prescribed singular parts. Divisors on a Riemann surface. Serre duality. Riemann-Roch theorem. Uniformization of genus zero and genus one surfaces. Algebraicity of compact Riemann surfaces. Compact Riemann surfaces are determined by function fields. Abel-Jacobi map embedding. Abel's theorem. Jacobi inversion theorem.
- If time permits: Belyi's theorem.

References

(a) Forster: Riemman surfaces.

(b) Farkas and Kra: Riemann surfaces.

E14: Several Complex Variables

- d-bar operator on $\mathbb{C}$, solution of d-bar equation for smooth data with compact support in $\mathbb{C}$. Harmonic functions in $\mathbb{C}$, harmonic conjugates, mean-value property, Poisson kernel and solution of the Dirichlet problem for the unit disc. Subharmonic functions in $\mathbb{C}$, maximum principle, various properties of subharmonic functions. Approximating subharmonic functions by smooth subharmonic functions. Definition of holomorphic functions and mappings of several complex variables. dbar operators in $\mathbb{C}^n$. Power series in several complex variables. Cauchy integral formula for holomorphic functions on a polydisc, local power series representation. Uniqueness principle. d-bar equation on $\mathbb{C}^n$: 1-forms, $(1,0)$-forms and $(0,1)$-forms. Solution of the d-bar equation for smooth data with compact support in $\mathbb{C}^n$, application to proving Hartogs phenomenon. Domains of convergence of power series, Reinhardt domains, logarithmically convex domains. Convergence of Taylor series on Reinhardt domains. Convexity with respect to a family of functions. Convex domains in $\mathbb{R}^n$, convex exhaustion functions, defining functions for strongly convex domains, continuity principle for convex domains. Definitions and basic properties of plurisubharmonic functions in $\mathbb{C}^n$. Definition of pseudoconvex domains in $\mathbb{C}^n$. Various equivalent characterizations of pseudoconvexity. The Levi form and Levi pseudoconvexity for domains with C^2-boundary, equivalence with pseudoconvexity. Defining functions for strictly Levi pseudoconvex domains. Holomorphic convexity. Domains of holomorphy. Domain holomorphically convex if and only if it is a domain of holomorphy. Domains of holomorphy are pseudoconvex. Hormander's solution of the d-bar equation on pseudoconvex domains. Unbounded operators on Hilbert spaces, adjoints, closed operators. Weak partial derivatives of locally integrable functions on $\mathbb{R}^n$. d-bar operators on functions and 1-forms. Weighted L^2-spaces of functions, $(0,1)$-forms and $(0,2)$-forms. Abstract criterion for existence of weak solutions to d-bar problem. Approximation in the graph norm of $L^2(0,1)$-forms by smooth $(0,1)$-forms with compact support. Existence of weak['6cf solutions to d-bar problem on pseudoconvex domains. Sobolev spaces, Sobolev embedding, and regularity of solutions to dbar problem on pseudoconvex domains. Solution of the Levi Problem: any pseudoconvex domain is a domain of holomorphy.

References

(a) Hormander: An Introduction to Complex Analysis in Several Variables.

(b) Krantz: Function Theory of Several Complex Variables.

E15: Quasiconformal mappings and Teichmuller theory

- Boundary behaviour of conformal mappings, Caratheodory's theorem (boundary locally connected if and only if the Riemann mapping extends continuously to the boundary). Conformal mapping between Jordan domains extends to a homeomorphism between closures. Definition of C^1 quasi-conformal mapping. Solution of Grotzsch problem on quasi-conformal mapping between two rectangles with minimum dilatation. Modulus of a quadrilateral in $\mathbb{C}$. Extremal len.gth of a path family in $\mathbb{C}$. Basic properties of extremal length. Calculation of extremal length for examples of path families in rectangles and annuli. Quasi-invariance of extremal length /un+der quasi-conformal mappings. Geometric definition of quasi-conformal mappings in terms of moduli of quadrilaterals. 1-quasi-conformal mappings are conformal. Analytic definition of quasi-conformal mappings in terms of absolute continuity on lines and weak derivatives. Equivalence of geometric and analytic definitions. Equicontinuity of uniformly quasi-conformal mappings. Convergence of quasi-conformal mappings. Solution of the Beltrami equation on the Riemann sphere. Definition of Teichmuller space of a Riemann surface, and Teichmuller metric. Existence of extremal mapping with minimum dilatation in a fixed homotopy class. Completeness of Tecihmuller metric. Teichmuller space of a complex torus is the hyperbolic plane. Holomorphic quadratic differentials on Riemann surfaces, natural parameters, associated flat metric and area form. Teichmuller mappings. Teichmuller's Uniqueness theorem: a Teichmuller mapping is uniquely extremal in its homotopy class. Teichmuller's homeomorphism from unit ball of holomorphic quadratic differentials to Teichmuller space. Fricke-Klein coordinates on Teichmuller space. Teichmuller's Existence theorem: each homotopy class contains a Teichmuller mapping. Teichmuller space homeomorphic to a cell of dimension $6g - 6$. The Schwarzian derivative, the Bers embedding and the complex structure on Teichmuller space.

References

(a) Ahlfors: Lectures on Quasiconformal Mappings.

(b) Lehto-Virtanen: Quasiconformal mappings in the plane.

(c) Imayoshi-Taniguchi: An Introduction to Teichmuller Spaces.

E16: Dynamical Systems

- Definition of dynamical systems and examples. Circle rotations. Expanding endomorphisms of the circle. Shifts and subshifts. Quadratic maps. Hyperbolic toral automorphisms. The horseshoe. The solenoid. Flows and differential equations. Suspension and cross-section. Chaos and Lyapunov exponents. Attractors. Limit sets and recurrence. Topological transitivity. Topological mixing. Expansiveness. Topological entropy, definition and calculation for some examples. Equicontinuity, distality and proximality. Measure preserving transformations and Poincarre recurrence theorem. Ergodicity and mixing. Invariant measures for continuous maps. Ergodic theorems. Unique ergodicity and Weyl's theorem. Weak mixing. Hyperbolic dynamics. Hyperbolic sets, stable and unstable subspaces. Pseudo-orbits and Anosov shadowing theorem. Hyperbolicity in terms of invariant cones. Stability of hyperbolic sets. Structural stability of Anosov diffeomorphisms. Graph transform and Hadamard-Perron theorem. Existence of local stable and unstable manifolds. Local product structure. Density of periodic points for Anosov diffeomorphisms. Density of stable and unstable manifolds, topological mixing of Anosov diffeomorphisms. Circle homeomorphisms, rotation number. Poincare classification of circle homeomorphisms. Denjoy's theorem on conjugacy to irrational rotation for C^2 circle diffeomorphisms. Complex dynamics. Fatou and Julia sets of rational maps. Local linearization near attracting fixed points, Bocher coordinates near superattracting fixed points. Finiteness of attracting and superattracting periodic orbits. Density of repelling periodic points in the Julia set. Backward iterates of points in the Julia set dense in the Julia set. Totally invariant components in the Fatou set. Hyperbolic rational maps and totally disconnected Julia sets.

References

(a) Garrett Stuck and Michael Brin: Introduction to Dynamical Systems.

(b) Anatole Katok and Boris Hasselblatt: Introduction to the Modern Theory of Dynamical Systems.

(c) John Willard Milnor: Dynamics in One Complex Variable.

E17: C^* algebraic K-theory

[Prerequisites: At least one course among Operator algebras, Basics of C^*-algebras, Advanced Functional Analysis]

- Homotopy classes of unitary elements, equivalence of projections, semigroup of projections, the K_0 group of a C^*-algebra and its functorial properties, half exactness, split exactness and stability of K_0, ordered K-theory, (definitions and a few easy computations only), K_1 group and its functorial properties, K_1 and determinants, The index map, higher K groups, the long exact sequence in K theory (statement with examples), Bott periodicity and its applications (sketch of proof if time permits), statement (without proof) and some easy applications of the Connes-Thom isomorphism and the Pimsner-Voiculescu 6-term exact sequence.

References

(a) N E Wegge-Olsen: K theory and C^*-algebras: A Friendly Approach.

(b) M. Rørdam, F. Larsen, N. Laustsen: An Introduction to K-theory for C^*-Algebras.

(c) Bruce Blackadar: K theory for Operator Algebras.

E18: Algebra III

- Modules over noncommutative rings: Noetherian and Artinian rings and modules. Modules of finite length. Krull-Schmidt theorem.
- Tensor product of modules and algebras: definitions, basic properties, right exactness, change of base.
- Semisimple rings and modules; Wedderburn-Artin theory.
- Nilradical and Jacobson radical; NAK lemma; Jacobson radical of an Artinian ring is nilpotent; Ring semi-simple if and only if Artinian with trivial radical; Artinian ring is Noetherian.
- Central simple algebras; Skolem-Noether theorem (from Lam's Noncommutative rings).
- Time permitting: Introduction to Quadratic forms; Witt's theorem; Clifford algebras (from Lam's Quadratic forms).

References

(a) T.Y. Lam, A First Course in Noncommutative Rings, Springer.

(b) T.Y. Lam, Introduction to Quadratic Forms, AMS.

(c) T. Hungerford, Algebra, Springer.

(d) P.M. Cohn, Further Algebra and Applications, Springer.

(e) S. Lang, Algebra, Springer.

(f) D.S. Dummit and R.M. Foote, Abstract Algebra (Part VI), John Wiley.

(g) TIFR notes on Semisimple rings and modules.

(h) N. Jacobson, Basic Algebra II, W.H. Freeman and Co (1985).

E19: Topics in Discrete Probability

- Review of discrete probability, First and Second Moment methods, Chernoff bounds and some applications.
- Percolation on lattices: Phase-transition phenomena, subcritical and supercritical phases, Uniqueness.
- Random graphs: Phase transition, Influences, Russo's formula and Sharp thresholds. Noise Sensitivity and Stability.
- Introduction to Markov chains and Martingales. Branching processes. Random walks and electrical networks, Uniform spanning trees.

References

(a) C. Garban and J. Steif: Noise Sensitivity of Boolean Functions and Percolation.

(b) N. Lanchier: Stochastic Modelling.

(c) Sebastien Roch: Modern Discrete Probability: A toolkit. (Notes).

(d) R. Lyons and Y. Peres: Probability on trees and networks.

(e) M. Barlow: Random walks and heat kernel on Graphs.

E20: Topics in Gaussian Processes [Prerequisites: Probability theory (C12)]

(*Note: It is recommended to do Brownian motion and diffusions (E51) in parallel.*)

- Review of Gaussian random variables. Slepian and Sudakov-Fernique inequalities. Variance bounds, Poincare inequality, Isoperimetric inequality, Log-sobolev inequality, Concentration and transport inequalities. Maxima of Gaussian processes. Majorizing measures and Generic chaining. Excursion probabilities. Hypercontractivity.
- Additional Topics (depending on time and audience interest): Geometry of Gaussian random fields. Stein's method. Introduction to Malliavin calculus.

References

(a) Ramon van Handel: Probability in high dimensions: Notes

(b) Manjunath Krishnapur's course on Gaussian processes: Notes

(c) R. J. Adler: Introduction to continuity, extrema and related topics for general Gaussian processes.

(d) R.J. Adler and J. E. Taylor: Random fields and Geometry.

(e) M. Talagrand: Upper and lower bounds for stochastic processes.

E21: Analysis on Graphs

- Review of Matrices and Graphs. Incidence matrix, Adjacency matrix and Laplace matrix of a graph. The Laplace operator on graphs. Random walks on graphs. Dirichlet Problem. Spectral properties. Eigenvalues and mixing time. Cheeger's inequality. Eigenvalues on infinite graphs. Estimates of the heat kernel. Carne-Varopoulous bound.

References

(a) R. B. Bapat: Graphs and Matrices.

(b) A. Grigor'yan.: Introduction to Analysis on Graphs.

(c) M. T. Barlow: Random Walks and Heat Kernel on Graphs.

(d) R. Lyons and Y. Peres: Probability on trees and networks.

E22: Algebraic Graph Theory

- Graph homomorphisms and automorphisms, Permutation groups (Orbits, primitivity and connectivity), Transitive graphs, Graphs and their spectra (adjacency algebra and spectrum of a graph, Perron-Frobenius theory, interlacing properties, invariant subspaces of an adjacency algebra), Cuts and Flows, Strongly regular graphs, Rank Polynomial and matroids.

References

(a) R. B. Bapat: Graphs and Matrices.

(b) N.L. Biggs, Algebraic Graph Theory.

(c) C. Godsil and G. Royle: Algebraic Graph Theory.

E23: Advanced Number Theory

- Review of finite fields; polynomial equations over finite fields: theorems of Chevalley and Warning; Quadratic Forms over prime fields.
- Review of the law of quadratic reciprocity.
- The ring of p-adic integers; the field of p-adic numbers; completion; p-adic equations and Hensel's lemma; Quadratic Forms with p-adic coefficients. Hilbert's symbol.

- Dirichlet series: abscissa of convergence and absolute convergence.
- Riemann Zeta function and Dirichlet L-functions. Dirichlet's theorem on primes in arithmetic progression. Functional equation and Euler product for L-functions.
- Modular forms and the modular group $SL(2,\mathbb{R})$. Eisenstein series. Zeros and poles of modular functions. Dimensions of the spaces of modular forms. The j-invariant and Picard's Theorem. L-function and Ramanujan's τ-function.

References

(a) J.P. Serre: A Course in Arithmetic, Springer-Verlag (1973).

(b) Z. Borevich and I. Shafarevich: Number Theory (chapter 1).

(c) K. Chandrasekharan: Introduction to Analytic Number Theory, Springer-Verlag (1968).

E24: Algebraic Number Theory

(*Note: A priori knowledge of "Commutative Algebra" is desirable.*)

- Algebraic numbers and algebraic integers; Brief review of integral extensions; Norm, trace and discriminant; Existence of integral basis. Dedekind domains, ideal class group. Minkowsky theory, finiteness of class group. Dirichlet unit theorem. Factoring of prime ideals on extensions, fundamental identity; Quadratic number fields (computation of class numbers, prime decomposition, Pell's equations). Hilbert's ramification theory (decomposition and inertia groups); Cyclotomic fields. Valuations, completions, local fields.

References

(a) G.J. Janusz: Algebraic Number Fields, (chapter 1-4), AMS (1996).

(b) D.A. Marcus: Number Fields, Springer-Verlag (1977).

(c) J. Neukirch: Algebraic Number Theory, Springer (1999).

(d) P. Ribenboim: Classical Theory of Algebraic Numbers, Springer Science and Business Media (2001).

(e) J. Esmonde and M. Ram Murty: Problems in Algebraic Number Theory, Springer (Indian reprint 2006).

(f) TIFR pamphlet on Algebraic Number Theory.

E25: Advanced Functional Analysis

- Brief introduction to topological vector spaces (TVS) and locally convex TVS. Linear Operators. Uniform Boundedness Principle. Geometric Hahn-Banach theorem and applications (Markov-Kakutani fixed point theorem, Haar Measure on locally compact abelian groups, Liapounovs theorem). Extreme points and Krein-Milman theorem.
- In addition, one of the following topics:
  (a) Geometry of Banach spaces: vector measures, Radon-Nikodym Property and geometric equivalents. Choquet theory. Weak compactness and Eberlein-Smulian Theorem. Schauder Basis.
  (b) Banach algebras, spectral radius, maximal ideal space, Gelfand transform.
  (c) Unbounded operators, Domains, Graphs, Adjoints, spectral theorem.

References

(a) N. Dunford and J. T. Schwartz, Linear operators. Part II: Spectral theory. Self adjoint operators in Hilbert space, Interscience Publishers, John Wiley (1963).

(b) Walter Rudin, Functional analysis, Second edition, International Series in Pure and Applied Mathematics. McGraw-Hill (1991).

(c) K. Yosida, Functional analysis, Springer (Indian reprint 2004).

(d) J. Diestel and J. J. Uhl, Jr., Vector measures, Mathematical Surveys (15), AMS (1977).

E26: Operator Algebras

- Definition and examples of Banach and C^* algebras, Gelfand theory for abelian Banach and C^* algebras, Spectral theorem for bounded normal operators on a Hilbert spaces.
- General noncommutative C^* algebras: basic properties, Positive elements and states, GNS theorem, ideals and quotients of C^* algebras, C^* algebra of compact operators.
- Definition of examples of von Neumann algebras, the double commutant theorem and Kaplansky density theorem, abelian von Neumann algebras and the L^∞-functional calculus.
- Time permitting: discussion on some concrete examples like AF algebras, group C^* algebras for discrete countable groups, Cuntz algebras, noncommutative tori etc.

References

(a) C^* Algebras by Examples, K R Davidson;

(b) Functional A Course in Functional Analysis, J B Conway

(c) C^* Algebra and Operator Theory, G. J. Murphy

(d) Fundamentals of the theory of operator algebras, volume I and II, R.V. Kadison and J. R. Ringrose

E27: Unbounded Operators

- Review of bounded operators on Hilbert spaces and spectral theorem for compact normal operators.
- 2) Definition and examples of Banach and C^* algebras, Gelfand theory for abelian Banach and C^* algebras, Spectral theorem for bounded normal operators on a Hilbert spaces.
- Unbounded operators: definition and examples; densely defined, closable and closed operators; graph of an unbounded operators; basic results involving these properties.
- Symmetric, self-adjoint and normal operators; Caley transform; spectral theorems for unbounded self adjoint and unbounded normal operators; examples and applications.

References

(a) Functional Analysis; W. Rudin

E28: Advanced Linear Algebra

- Majorization and doubly stochastic matrices. Matrix Decomposition Theorems (Polar, QR, LR, SVD etc.) and their applications. Perturbation Theory.
- Nonnegative matrices and their applications. Wavelets and the Fast Fourier Transform. Basic ideas of matrix computations.

References

(a) R. Bhatia, Matrix Analysis, GTM (169), Springer (Indian reprint 2004).

E29: Markov Chains

- Finite State Markov Chains. Examples, Classification of States, Stationary Distribution.
- Random walk on Finite Groups. Connection to electrical networks. Recurrence and Transience of Random walks.
- Branching chain, progeny distribution and progeny generating function, extinction probability, geometric growth in the super-critical case.
- Rates of convergence to stationarity, Dirichlet Form and Spectral gap methods, Some Coupling methods with applications, Cheeger's inequality.
- Poisson Processes, Continuous time Markov Chains, Birth-and-death processes.

References

(a) S. M. Ross, Stochastic processes, John Wiley (1996).

(b) R. N. Bhattacharya and E. C. Waymire, Stochastic processes with applications.

(c) E. Gine, R. Grimmett and L. Saloff-Coste, Lectures on probability theory and statistics, Springer-Verlag (1997).

(d) L. Levine, Y. Peres and E. Wilmer: Markov Chains and Mixing times.

(e) D. Aldous and J. A. Fill: Reversible Markov Chains and Random Walk on Graphs.

E30: Ergodic Theory

- Measure preserving systems; examples. Hamiltonian dynamics and Liouville's theorem, Bernoulli shifts, Markov shifts, Rotations of the circle, Rotations of the torus, Automorphisms of the Torus, Gauss transformations, Skew-product.
- Poincare Recurrence lemma. Induced transformation: Kakutani towers: Rokhlin's lemma. Recurrence in Topological Dynamics, Birkhoff's Recurrence theorem.
- Ergodicity, Weak-mixing and strong-mixing and their characterisations. Ergodic theorems of Birkhoff and Von Neumann. Consequences of the ergodic theorems. Invariant measures on compact systems. Unique ergodicity and equidistribution. Weyl's theorem.
- Isomorphism problem; conjugacy, spectral equivalence.
- Transformations with discrete spectrum, Halmos von Neumann theorem.
- Entropy. The Kolmogorov-Sinai theorem. Calculation of Entropy. Shannon-McMillan-Breiman Theorem.

- Flows. Birkhoff's Ergodic Theorem and Wiener's Ergodic Theorem for flows. Flows built under a function.

References

(a) Peter Walters, An introduction to Ergodic Theory, GTM (79), Springer (Indian reprint 2005).

(b) Patrick Billingsley, Ergodic theory and information, Robert E. Krieger Publishing Co. (1978).

(c) M. G. Nadkarni, Basic ergodic theory, TRIM 6, Hindustan Book Agency (1995).

(d) H. Furstenberg, Recurrence in ergodic theory and combinatorial number theory, Princeton University Press (1981).

(e) K. Petersen, Ergodic theory, Cambridge Studies in Advanced Mathematics (2), Cambridge University Press (1989).

E31: Topology III

- CW-complexes, cellular homology, comparison with singular theory, computation of homology of projective spaces.
- Definition of singular cohomology, axiomatic properties, statement of universal coefficient theorem for cohomology. Betti numbers and Euler characterisitic. Cup and cap product, Poincare duality. Cross product and statement of Kunneth theorem. Degree of maps with applications to spheres.
- Definition of higher homotopy groups, homotopy exact sequence of a pair. Definition of fibration, examples of fibrations, homotopy exact sequence of a fibration, its application to computation of homotopy groups. Hurewicz homomorphism, The Hurewicz theorem. The Whitehead Theorem.

References

(a) A. Hatcher, Algebraic Topology, Cambridge University Press (2002).

(b) M. J. Greenberg and J.R. Harper, Algebraic topology: A First Course, Benjamin/Cummings (1981).

(c) E. Spanier, Algebraic Topology, Springer-Verlag (1982).

(d) J.W. Vick, Homology Theory: an introduction to algebraic topology, Springer (1994).

(e) J. R. Munkres, Elements of algebraic topology, Addison-Wesley (1984).

(f) G.E. Bredon, Topology and Geometry, Springer (Indian reprint 2005).

E32: Topology IV

[Prerequisite: Differential Geometry I]

- Smooth manifolds, differential forms on manifolds, integration on manifolds, Stoke's theorem, computation of cohomology rings of projective spaces, Borsuk-Ulam theorem.
- Degree, linking number and index of vector fields, the Poincare-Hopf theorem.
- Definition and examples of principal bundles and fibre bundles, clutching construction, description of classification theorem (without proof).

References

(a) R. Bott and L. W. Tu, Differential forms in algebraic topology, GTM (82), Springer-Verlag (1982).

(b) Ib H. Madsen and J. Tornehave, From Calculus to Cohomology: De Rham Cohomology and Characteristic Classes, Cambridge Univ Press (1997).

(c) F. W. Warner,t Foundations of differentiable manifolds and Lie groups, GTM (94), Springer-Verlag (1983).

(d) D. Husemoller, Fibre Bundles, Springer-Verlag (1993).

(e) N. Steenrod, The Topology of Fibre Bundles, Princeton Univ Press (1999)

E33: Differential Geometry II

[Prerequisite: Differential Geometry I]

- A quick review of tensors, alternating forms, manifolds, immersion, submersion and submanifolds.
- Tangent bundle, vector bundles, vector fields, flows and the fundamental theorem of ODE. Riemann metrics, Riemannian connection on Riemannian manifolds. Parallel transport, geodesics and geodesic completeness, the theorem of Hopf-Rinow.
- Time permitting: Gauss-Bonnet theorem.

References

(a) F. W. Warner, Foundations of differentiable manifolds
and Lie groups, GTM (94), Springer-Verlag (1983).

(b) S. Helgason, Differential geometry, Lie groups, and symmetric space, Graduate Studies in Mathematics (34), AMS (2001).

(c) W.M. Boothby, An Introduction to Differentiable Manifolds and Riemannian Geometry, Academic Press (1975); Elsevier (2008).

(d) J.M. Lee, Riemannian Manifolds: An Introduction to Curvature, GTM (176), Springer (1997).

E34: Elliptic Curves

[Prerequisites: algebraic number theory (could be simultaneous), algebraic geometry (could be simultaneous)]

- Algebraic curves, divisors, Riemann-Roch theorem.
- Definition of elliptic curves, Weierstrass form, isogeny, Tate module, Weil pairing, Endomorphism ring.
- Elliptic functions and integrals, Elliptic curves over complex numbers, Uniformization.
- Elliptic curves over finite fields, Weil conjectures, Hasse invariant.
- Elliptic curves over local fields, Minimal Weierstrass equation, Torsion, Good and bad reduction and Neron-Ogg-Shafarevich criterion for good reduction.
- Elliptic curves over global fields, weak Mordell-Weil, Kummer pairing, Mordell-Weil theorem over $\mathbb{Q}$.
- If time permits: Heights on projective spaces and elliptic curves and Mordell-Weil theorem; Nagell-Lutz theorem.

References

(a) J. Silverman, Arithmetic of elliptic curves (chapters 2,3,5,6,7 and sections 8.1 to 8.4), GTM 106, Springer-Verlag (1986).

(b) N. Koblitz, Elliptic Curves and Modular Forms, GTM 97, Springer-Verlag 1984.

(c) J.W.S. Cassels, Lectures on Elliptic Curves, Cambridge Uni Press 1991.

(d) D. Husemoller, Elliptic Curves, Springer Science and Business Media, Vol.111, 1987.

E35: Convex Geometry

- Geometric foundations: combinatorial properties, support and separation theorems, extremal representations.
- Convex functions.
- The Brunn-Minkowski Theory: basic functionals of convex bodies, mixed volumes, geometric (isoperimetric) inequalities.
- Surface area measures and projection functions.
- Integral geometric formulas.

References

(a) P. Gruber. Convex and Discrete Geometry.

(b) R. Schneider. Convex Bodies: the Brunn-Minkowski theory.

(c) D. Hug and W. Weil: Lectures in Convex Geometry.

E36: Linear Algebraic Groups

[Prerequisites: Commutative Algebra, Lie Algebras (could be simultaneous), Algebraic Geometry (could be simultaneous)]

- Review of background commutative algebra and algebraic geometry (as in chapter 1 of Humphreys's book or chapter 1 of Springer's book). Definition of affine algebraic groups and homomorphisms over algebraically closed fields, examples.
- Orbit-closures under actions, linearity of affine groups. Lie algebra of an algebraic group and adjoint representation. Homogeneous spaces and quotients, Chevalley's theorem. Correspondence between groups and Lie algebras. Jordan decomposition, Commutative linear algebraic groups, diagonalizable groups and algebraic tori. Definition of weights and roots, Weyl group. Unipotent groups, Lie-Kolchin theorem, Structure theorem for connected solvable groups. Definition of reductive and semisimple groups, Borel subgroups, parabolic subgroups. Basic facts on complete varieties, Borel's fixed point theorem. Conjugacy of maximal tori, Nilpotency of Cartan subgroups. Density theorem and connectedness of centralizers of tori. Normalizer theorem for parabolic subgroups. Regular and singular tori, Structure theorem for groups of semisimple rank one. Structure theorem for reductive groups, Bruhat decomposition, semisimple groups. Tits system, standard parabolic subgroups, simplicity proof.

- If time permits, mention (without proof): Representations and classification of semisimple groups and statements for general fields.

References

(a) J. E. Humphreys, Linear algebraic groups, (chapters 1 to 10), GTM, Springer-Verlag (1975).

(b) T. A. Springer, Linear algebraic groups, (chapters 1 to 8), Progress in Mathematics, Birkhäuser (1998).

(c) R. Steinberg, Conjugacy classes in algebraic groups, (Chapters 1 and 2 just as reference, some proofs are not given), Lecture Notes in Mathematics (366), Springer-Verlag (1974).

E37: Mathematical Logic

- Syntax of First-Order Logic: First Order Languages, Terms and Formulas of a First Order language, First Order Theories.
- Semantics of First-Order Languages: Structures of First-Order Languages, Truth in a Structure, Model of a Theory.
- Propositional Logic: Tautologies and Theorems of propositional Logic, Tautology Theorem.
- Proof in First Order Logic, Metatheorems of a first order theory, e.g., theorems on constants, equivalence theorem, deduction and variant theorems etc., Consistency and Completeness, Lindenbaum Theorem.
- Henkin Extension, Completeness theorem, Extensions by definition of first order theories, Interpretation theorem.
- Model Theory: Embeddings and Isomorphisms, Löwenheim-Skolem Theorem, Compactness theorem, Categoricity, Complete Theories.
- Recursive functions, Arithmatization of first order theories, Decidable Theory, Representability, Godel's first Incompleteness theorem.

References

(a) S. M. Srivastava, A Course on Mathematical Logic, Springer (2013).

(b) J. R. Shoenfield, Mathematical logic, Addison-Wesley (1967).

E38: Set Theory

Either (A) or (B):

(A) *Descriptive Set Theory:*

- A quick review of elementary cardinal and ordinal numbers, transfinite induction, induction on trees, Idempotence of Souslin operation.
- Polish spaces, Baire category theorem, Transfer theorems, Standard Borel spaces, Borel isomorphism theorem, Sets with Baire property, Kuratowski-Ulam Theorem. The projective hierarchy and and its closure properties.
- Analytic and coanalytic sets and their regularity properties, separation and reduction theorems, Von Neumann and Kuratowski-Ryll Nardzewskis selection theorems, Uniformization of Borel sets with large and small sections. Kondos uniformization theorem.

References

(a) S. M. Srivastava, A course on Borel sets, GTM (180), Springer-Verlag (1998).

(b) A. S. Kechris, Classical descriptive set theory, GTM (156), Springer-Verlag (1995).

(B) *Axiomatic Set Theory:*

- A naive review of cardinal and ordinal numbers including regular and singular cardinals, some large cardinals like inaccessible and measurable cardinals. Martins Axiom and its consequences. Axiomatic development of set theory upto foundation axiom, Class and Class as models, relative consistency, absoluteness, Reflection principle, Mostowski collapse lemma, non-decidability of large cardinal axioms, Godel's second incompleteness theorem, Godel's constructible universe, Forcing lemma and independence of CH.

References

(a) S. M. Srivastava, A course on Borel sets, GTM (180), Springer-Verlag (1998).

(b) K. Kunen, Set theory. An introduction to independence proofs, Studies in Logic and the Foundations of Mathematics (102), North-Holland Publishing Co. (1980).

(c) T. Jech, Set theory, Academic Press (1978).

E39: Game Theory

(*Note: The course may be combined with the course "Game Theory-I" of the MSQE programme (see (1) in 3.2) or the course "Theory of games and statistical decisions" of M.Stat. 2nd year (see (22) in 3.3) at ISI.*)

- Non-Cooperative Games: Games in normal form. Rationalizability and iterated deletion of never-best responses. Nash equilibrium: existence, properties and applications. Two-person Zero Sum Games. Games in extensive form: perfect recall and behavoiour strategies. Credibility and Subgame. Perfect Nash equilibrium. Bargaining. Repeated Games; Folk Theorems.
- Introduction to Cooperative Games (TU Games).

References:

(a) A. R. Karlin and Y. Peres. Game Theory, Alive (2016).

(b) T. Ferguson. A Course in Game Theory (2020).

(c) D. Easley and J. Kleinberg. Networks, Crowds, and Markets (2010).

(d) M. Maschler, E. Solan and S. Zamir. Game Theory (2013).

(e) M. J. Osborne and A. Rubinstein. A Course in Game Theory (1994).

(f) Roger B. Myerson. Game Theory: Analysis of Conflict (1991).

(g) D. Fudenberg and J. Tirole. Game Theory (1991).

(h) Y. Narahari. Game Theory and Mechanism Design (2014).

E40: Automata, Languages and Computation

- Automata and Languages: Finite automata, regular languages, regular expressions, equivalence of deterministic and non-deterministic finite automata, minimisation of finite automata, closure properties, Kleene's theorem, pumping lemma and its applications, Myhill-Nerode theorem and its uses. Context-free grammar, context-free languages, Chomsky normal form, closure properties, pumping lemma for CFL, pushdown automata.
- Computability: Computable functions, primitive and partial recursive functions, universality and halting problem, recursive and recursively enumerable sets, parameter theorem, diagonalisation and reducibility, Rice's theorem and its applications, Turing machines and its variants, equivalence of different models of computation and Church-Turing thesis.

- Complexity: Time complexity of deterministic and non-deterministic Turing machines, P and NP, NP-completeness, Cook's theorem: other NP-complete problems.

References

(a) N. Cutland, Computability. An introduction to recursive function theory, Cambridge University Press (1980).

(b) M. D. Davis, Ron Sigal and E. J. Weyuker, Computability, complexity, and languages. Fundamentals of theoretical computer science, Academic Press (1994).

(c) J. E. Hopcroft and J. D. Ullman, Introduction to automata theory, languages, and computation, Addison-Wesley (1979).

(d) H. R. Lewis and C. H. Papadimitriou, Elements of the theory of computation, Prentice-Hall (1981).

(e) S. M. Sipser, Introduction to the theory of computation, PWS Pub Co, NY (1999).

(f) M. R. Garey and D. S. Johnson, Computers and intractability. A guide to the theory of NP-completeness, W. H. Freeman and Co. (1979).

E41: Advanced Fluid Dynamics

- Inviscid incompressible fluid: Two dimensional motion, stream function, complex potential and velocity, sources, sinks. Doublets and their images. Circle theorem, Blasiuss theorem, Kutta-Jokowaski theorem. Axi-symmetric motion, Stokes stream function. Image of a source and a sink with respect to a sphere. Vortex motion, vortex lines and filaments, systems of vortices, rectilinear vortices, vortex pair and doublets. A single infinite row of vortices, Karmans vortex sheet.
- Linearised gravity waves, progressive waves in deep and shallow water, stationary waves, energy and group velocity, long waves and their energy, capillary waves.
- Inviscid compressible fluid: First and second law of thermodynamics, polytropic gas and its entropy, adiabatic and isentropic flow, propagation of small disturbances. Mach number, Mach cone, irrotational motion, Bernoulli's' Equation, pressure, density and temperature in terms of Mach number. Area velocity relations in one-dimensional flow, concept of subsonic and supersonic flows. Normal shock-wave, Rankine-Hugonoit and Prandtl's' relations in case of a plane shock wave.

- Viscous incompressible fluid: Equations of motion of a viscous fluid, Reynold's number, circulation in a viscous liquid, Flow between parallel plates, flow through pipes of circular, elliptic and annular section under constant pressure gradient. Prandtl's concept of boundary layer.

References

(a) L. M. Milne-Thomson, Theoretical hydrodynamics, Macmillan (1960).

(b) L. D. Landau and E. M. Lifshitz, Fluid mechanics, Course of Theoretical Physics, Vol. 6 Pergamon Press (1959).

(c) H. Lamb, Hydrodynamics, Cambridge University Press (1993).

(d) W. H. Besant and A. S. Ramsey, A treatise of Hydro-mechanics, Part II, ELBS.

(e) P. K. Kundu, Fluid mechanics, Academic Press.

E42: Quantum Mechanics I

- (i) Physical Basis of Quantum Mechanics. (ii) Old Quantum theory. (iii) Uncertainty, Complimentarity and Duality. (iv) Measurement problems. (v) Heisenberg and Schrodinger representation.
- (i) Schrodinger wave equation (ii) Perturbation theory.
- Problem of two or more degrees of freedom without spherical symmetry; Stark effect.
- Angular momentum, $SU(2)$ algebra.
- Three-dimensional Schrodinger equation. Problems with spherical symmetry. Harmonic Oscillator.
- Scattering problem , differential cross section, phase shift, variational principle, SW transformation, Regge poles.
- WKB approximation.
- Particles with spin, Pauli matrices, Pauli-Schrodinger equation. Two and three body problems. Hydrogen atom in electric and magnetic field.
- Quantum Statistics.

References

(a) L.I. Schiff, Quantum Mechanics.

(b) J.J. Sakurai, Modern Quantum Mechanics.

(c) L. D. Landau and E. M. Lifshitz, Quantum mechanics: non-relativistic theory, Course of Theoretical Physics Vol 3, Pergamon Press Ltd (1958).

(d) L.M. Falicov, Group theory and its physical applications, Univ of Chicago Press (1966).

E43: Quantum Mechanics II

- Non stationary problems. Relativistic Dirac equation, Spinors. Scattering by a central force. Radiation theory. Quantization of Schrodinger field. Born approximation. Compton effect (Klein Nishina formula). Bremsstrahlung. Symmetry and conservation laws. Quantum Probability and quantum Statistics. Supersymmetric Quantum Mechanics, SWKB. Path integral method.

References

(a) L.I. Schiff, Quantum Mechanics.

(b) P.A.M. Dirac, The Principles of Quantum Mechanics, Oxford, Clarendon Press (1947).

(c) P. A. M. Dirac, Spinors in Hilbert space, Plenum Press (1974).

(d) M. E. Rose, Elementary theory of angular momentum, John Wiley.

(e) R. P. Feynman and A. R. Hibbs, Quantum Mechanics and Path integrals.

(f) L. D. Landau and E. M. Lifshitz, Statistical physics, Course of Theoretical Physics. Vol. 5. Pergamon Press Ltd (1958).

(g) S. Flugge, Practical quantum mechanics, Springer-Verlag (1999).

(h) H. Weyl, The theory of groups and Quantum Mechanics.

E44: Analytical Mechanics

- Generalised coordinates, Lagrange's' Equation. Examples of Lagrange's' equation. Conservation laws. Motion in a central field. Collision of particles. Small Oscillations. Rotating Coordinate systems. Inertial forces. Dynamics of a rigid body. Hamiltonian Mechanics.

References

(a) I. Arnold, Mathematical methods of classical mechanics, GTM (60), Springer-Verlag (1978).

(b) R. Abraham and J. E. Marsden, Foundations of mechanics, Second edition, Benjamin/Cummings (1978).

E45: Representations of Locally Compact Groups

(a) Topological Groups, basic properties like subgroups, quotients and products, fundamental systems of neighbourhoods, open subgroups, connectedness and compactness. Existence of Haar measure on locally compact groups, properties of Haar measures.

(b) Group actions on topological spaces, the space X/G in the topological as also in the analytical case assuming regularity conditions of the group action.

(c) Representation of a locally compact group on a Hilbert space, the associated representation of group algebra, invariant subspaces and irreducibility, Schur's lemma.

(d) Compact groups: Unitarity of finite dimensional representations, Peter-Weyl theory, Representations of $SU(2, \mathbb{C})$, Representation of a finite group.

(e) Induced representation and Frobenius reciprocity theorem, Representations of Heisenberg groups and of Euclidean motion group, Principal series representations of $SL(2, \mathbb{R})$.

References

(a) P. J. Higgins, *Introduction to topological groups*, LMS Lecture Notes Series (15), Cambridge University Press (1974).

(b) L. H. Loomis: *An introduction to abstract harmonic analysis*, D. Van Nostrand (1953).

(c) G. B. Folland: *A course in abstract harmonic analysis*, Studies in Advanced Mathematics. CRC Press (1995).

E46: Abstract Harmonic Analysis

[Prerequisite: Fourier Analysis]

- Banach Algebras and spectral theory: Banach Algebras, Gelfand theory, spectral theorem.
- Generalities on Locally compact groups: Haar measure, modular function and convolution.
- Basic representation theory: Unitary representation of groups, Schur's lemma, Positive definite functions and GNS construction.

- Analysis on Locally compact abelian groups: Fourier transform and the dual group, Pontrajin Duality, Bochner's theorem, Plancherel theorem.
- If time permits: Peter Weyl theorem and analysis on compact groups.

References

(a) A course in abstract Harmonic Analysis- G. B. Folland.

(b) Principles of Harmonic Analysis- A. Deitmar, S. Echterhoff.

(c) Fourier Analysis on groups- W. Rudin.

(d) Fourier Analysis on Number Fields- Dinakar Ramakrishnan, Robert J. Valenza.

E47: Wavelet Analysis

- *Review of Fourier analysis*: Fourier series, Parseval's identity, Fourier transform of L^1- and L^2- functions, Plancherel theorem.
- *Multiresolution analysis (MRA)*: Construction of wavelets from an MRA, construction of compactly supported wavelets.
- *Some important wavelets*: Haar wavelet, Strömberg wavelet, Lemarié-Meyer wavelets, spline wavelets, band-limited wavelets and wavelet sets.
- *Frames*: The reconstruction formula for frames, Balian-Low theorem for frames, frames form traslations and dilations (wavelet frames).
- *Representation of functions by wavelets*: Convergence of wavelet expansions in $L^p(\mathbb{R})$, pointwise convergence of wavelet expansions, wavelets as unconditional bases for $L^p(\mathbb{R})$.
- *Optional topics*:
  (a) Wavelets on $\mathbb{T}$. Periodization of wavelets on $\mathbb{R}$, orthonormal bases of piecewise linear continuous functions for $L^2(\mathbb{T})$, orthonormal bases of periodic splines.
  (b) Characterization of various functions associated with wavelets. Characterization of scaling functions, characterization of wavelets, characterization of MRA-wavelets.
  (c) Wavelets in higher dimensions. Tensor product of one-dimensional wavelets, MRA in higher dimensions associated with dilation matrices, construction of Haar wavelets.

References

(a) E. Hernández and G. Weiss, A first course on wavelets, CRC Press, Boca Raton, 1996.

(b) P. Wojtaszczyk, A mathematical introduction to wavelets, Cambridge University Press, 1997.

(c) Y. Meyer, Wavelets and operators, Cambridge University Press, 1992.

(d) I. Daubechies, Ten lectures on wavelets, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, 1992.

E48: Basics of C^* Algebras

- Quick review of bounded operators on Hilbert spaces, Banach and C^*-algebras, Gelfand's Theorem for abelian C^*-algebras. Spectral Theorem for bounded normal operators.
- Noncommutative states and representations, Gelfand-Neumark rep- resentation theorem.
- Von-Neumann Algebras; Projections, Double Commutant theorem, L^∞ functional Calculus.
- Time Permitting: Some concrete examples like Cuntz algebras, AF algebras, group C^*-algebras etc.

References

(a) W. Arveson, An invitation to C^*-algebras, GTM (39), Springer-Verlag (1976).

(b) N. Dunford and J. T. Schwartz, Linear operators. Part II: Spectral theory. Self adjoint operators in Hilbert space, Interscience Publishers John Wiley (1963).

(c) R. V. Kadison and J. R. Ringrose, Fundamentals of the theory of operator algebras. Vol. I. Elementary theory, Pure and Applied Mathematics (100), Academic Press (1983).

(d) V. S. Sunder, An invitation to von Neumann algebras, Universitext, Springer-Verlag (1987).

(e) K. R. Davidson, C^* Algebras by example, Fields Institute Monograph (also available in TRIM series).

E49: Martingale Theory

[Prerequisite: Probability Theory]

(*Note: This has the same syllabus as the M-Stat course with the same name.*)

- Absolute continuity and singularity of measures. Hahn-Jordon decomposition, Radon-Nikodym Theorem, Lebesgue decomposition. Conditional expectation - Definition and Properties. Regular conditional probability, proper RCP. Regular conditional distribution.
- Discrete parameter martingales, sub-and super-martingales. Doob's Maximal Inequality, Upcrossing inequality, martingale convergence theorem, Lp inequality, uniformly integrable martingales, reverse martingales, Levy's upward and downward theorems. Stopping times, Doob's optional sampling theorem. Discrete martingale transform, Doob's Decomposition Theorem.
- Applications of martingale theory: Azuma-Hoeffding Inequality and some applications. SLLN for i.i.d. random variables. Infinite products of probability spaces, Hewitt-Savage 0-1 Law. Finite and infinite exchangeable sequence of random variables, de Finetti's Theorem. SLLN for U-Statistics for exchangeable data.
- Introduction to continuous parameter martingales: definition, examples and basic properties.
- (If time permits) Martingale Central Limit Theorem and applications.

References :

(a) Y. S. Chow and H. Teicher: Probability Theory

(b) Leo Breiman: Probability Theory

(c) Jacques Nevue: Discrete Parameter Martingales

(d) P. Hall , C. C. Heyde: Martingale Limit Theory and its Application

(e) R. Durret: Probability Theory and Examples

(f) P. Billingsley: Probability and Measures

E50: Theory of Large Deviations

[Prerequisite: Probability Theory]

(*Note: This has the same syllabus as the M-Stat course with the same name*.)

- Introduction to large deviations.
- Sanov's theorem and Cramer's theorem for finitely supported random variables.
- General notion of large deviation principle on Polish spaces: Laplace principle, Varadhan's lemma, weak large deviation principle, exponential tightness, goodness of rate function, contraction principle, Bryc's lemma.

- Cramer's theorem for general random variables and vectors.
- Exponential tightness of (a) sample averages of i.i.d. Banach space valued random variables and (b) empirical measures of i.i.d. Polish space valued random variables.
- Cramer's theorem on locally convex separable Hausdorff topological vector spaces.
- Large deviations of Brownian paths: Schilder's theorem.
- Sanov's theorem on Polish spaces: Donsker-Varadhan variational formula. Gartner and Ellis theorem.

References

(a) Large Deviations Techniques and Application by A. Dembo and O. Zeitouni

(b) Large Deviations by Deuschel and Stroock

(c) Large Deviations by Hollander

(d) Large Deviations and Applications by S. R. S. Varadhan

(e) A Weak Convergence Approach to the Theory of Large Deviations by P. Dupuis and T. Ellis.

E51: Brownian Motion and Diffusions

[Prerequisites: Probability Theory, Martingale Theory (could be simultaneous)]

(*Note: This has the same syllabus as the M-Stat course with the same name*.)

- Introduction to Brownian Motion, Kolmogorov Consistency theorem, Kolmogorov Continuity theorem, Construction of BM. Basic Martingale Properties and path properties -including Holder continuity and non-differentiability. Quadratic variation.
- Markov Property and strong Markov property of BM, reflection principle, Blumenthal's 0-1 law. Distributions of first passage time and of running maximum of BM.
- Brownian Bridge as BM conditioned to return to zero.
- Ito Integral with respect to BM, properties of Ito integral. Ito formula, Levy characterization, representation of continuous martingales of Brownian filtration.

- Continuous path Polish space-valued markov processes, Feller processes, Associated semigroup operators, resolvent operators and generators on the Banach space of bounded continuous functions. Generator of BM.
- Ito diffusions, Markov property of Ito diffusions, Generators of Ito diffusions.

References

(a) K. Ito: TIFR Lecture Notes on Stochastic Processes

(b) I. Karatzas , S. E. Shreve: Brownian Motion and Stochastic Calculus

(c) D. Freedman: Brownian Motion and Diffusion

(d) H. P. Mckean: Stochastic Integrals

E52: Weak Convergence and Empirical Processes

[Prerequisite: Probability Theory]

(*Note: This has the same syllabus as the M-Stat course with the same name.*)

- Probability measures on metric spaces. Weak convergence of probability measures on metric spaces. Portmanteau theorem. Convergence determining classes. Continuity theorem. Prohorov's theorem. Levy-Prohorov metric, Skorohod representation theorem.
- Weak convergence on $C(0,1)$, Arzela-Ascoli theorem, sufficient conditions for weak convergence on $C(0,1)$.
- Construction of Weiner measure on $C(0,1)$, Donsker's theorem, Application of continuity theorem to derive distributions of certain functionals of BM. Kolmogorov-Smirnov statistics. Weiner measure on $C(0,\infty)$.
- $D(0,1)$, Skorohod topology on $D(0,1)$, compactness on $D(0,1)$. Weak convergence of probability measures on $D(0,1)$. Empirical distribution functions, Donsker's Theorem on $D(0,1)$.
- Vapnik-Chervonenkis Theory in Empirical processes: Glivenko-Cantelli classes, Donsker classes, Vapnik-Chervonenkis classes, Shattering and VC-index, VC inequality with applications to convergence results.

References

(a) P. Billingsley: Weak Convergence of Probability Measures

(b) K. R. Parthasarathy: Probability measures on Metric Spaces

(c) D. Pollard: Empirical Processes